ON GROUPS OF ORDER 3^m AND CLASS 3

BY

JEWELL EMMA SCHUBERT

B.S., Northwestern University, 1936

M.S., University of Illinois, 1944

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN
MATHEMATICS IN THE GRADUATE COLLEGE
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R. UNIV. BIBLIOTHEEK LEIDEN

ON GROUPS OF ORDER 3^m AND CLASS 3

1. A classification of groups of finite order will depend upon a classification of prime-power groups as indicated by Sylow's theorem. With every group of order p^m there is associated one or more groups, e.g., subgroups and quotient groups, consisting entirely of elements of period p, each of which is completely determined by the original group. Hence the original group cannot be identified unless these groups all of whose elements are of period p can be identified.

Let G be such a group generated by n operators $u_1, \ldots, u_n$. If $p=2$, G is abelian. W. Burnside showed that if $p=3$, G is finite,[1] while F. Levi and B. L. Van der Waerden showed that such a group is of class 3 when no further relations exist.[2] Because every element in G is of period 3 and permutes with all of its transforms, it may be shown that the commutator subgroup K is abelian of type 1, 1, . . . , 1. Furthermore it may be shown that the commutator of a commutator and any element of G is in the central C. Hence the commutator subgroup of G/C is in the central of G/C, so that G/C is metabelian. Consequently G itself is of class 3.

For every integer $n \geqslant 3$ there exists a group G of class 3 generated by n operators where the only relations are $a_i^3 = I$ for every element $a_i \neq I$ in G. From the n generators $n(n-1)$ first commutators $u_i^{-1} u_j^{-1} u_i u_j = s_{ij}$ may be formed. Since $s_{ij} = s_{ji}^{-1}$ only $\left(\frac{n}{2}\right)$ of them are independent. Since K is abelian, the commutators of first commutators are the identity. $n\left[\frac{n(n-1)}{2}\right]$ second commutators may be formed as commutators of s_{ij} and u_k. Each s_{ij} permutes with u_i and u_j so that at most $n\left[\frac{n(n-1)}{2}\right] - n(n-1) = \frac{n(n-1)(n-2)}{2}$ of them $\neq I$. Not all of these are distinct since

$$s_{ij}^{-1} u_k^{-1} s_{ij} u_k = s_{jk}^{-1} u_i^{-1} s_{jk} u_i = s_{ki}^{-1} u_j^{-1} s_{ki} u_j.$$

Consequently at most $\left(\frac{n}{3}\right)$ of the second commutators are independent. Since the second commutators are in the central no further commutators may be obtained. Altogether then there is a basis of

[1] W. Burnside, "On an Unsettled Question in the Theory of Discontinuous Groups," Quarterly Journal of Mathematics, vol. 33 (1902), pp. 230-238.

[2] F. Levi u. B. L. Van der Waerden, "Über eine besondere Klasse von Gruppen," Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität, vol. 9 (1933), pp. 154-158.

$n+\binom{n}{2}+\binom{n}{3}$ operators in G, indicating the manner in which Levi and Van der Waerden established[3] the order of G to be $3^{\binom{n}{1}+\binom{n}{2}+\binom{n}{3}}$ where the orders of the central and the commutator subgroup are $3^{\binom{n}{3}}$ and $3^{\binom{n}{2}+\binom{n}{3}}$.

Further groups of class 3 generated by n operators are defined by imposing additional relations on the elements of G and are therefore determined as quotient groups of G.[4] Thus the problem of their determination resolves itself into the investigation of the types of invariant subgroups of G.

All the groups of class 3 generated by 4 operators are obtained in this fashion and listed in §2, which also illustrates the method used in examining subgroups of order 3 of the central when $n=5$ and $n=6$.

We note here that if $n=3$, G is of order 3^7 with a commutator subgroup of order 3^4 containing a central which is only of order 3, so that no proper subgroup of the central exists which would lead to a new group of class 3.

2. Let G be the group generated by 4 operators u_1, u_2, u_3, u_4, all of whose elements $\neq I$ are of period 3. Then G is of order 3^{14} with a commutator subgroup K of order 3^{10} and a central C of order 3^4. The relations aside from those expressing permutability of operators are

$$u_2^{-1}u_1u_2=u_1s_1 \qquad u_3^{-1}u_2u_3=u_2s_4 \qquad u_4^{-1}u_3u_4=u_3s_6$$
$$u_3^{-1}u_1u_3=u_1s_2 \qquad u_4^{-1}u_2u_4=u_2s_5$$
$$u_4^{-1}u_1u_4=u_1s_3$$

and

$$u_3^{-1}s_1u_3=s_1t_1 \qquad u_2^{-1}s_2u_2=s_2t_1^{-1} \qquad u_2^{-1}s_3u_2=s_3t_2^{-1}$$
$$u_4^{-1}s_1u_4=s_1t_2 \qquad u_4^{-1}s_2u_4=s_2t_3 \qquad u_3^{-1}s_3u_3=s_3t_3^{-1}$$
$$u_1^{-1}s_4u_1=s_4t_1 \qquad u_1^{-1}s_5u_1=s_5t_2 \qquad u_1^{-1}s_6u_1=s_6t_3$$
$$u_4^{-1}s_4u_4=s_4t_4 \qquad u_3^{-1}s_5u_3=s_5t_4^{-1} \qquad u_2^{-1}s_6u_2=s_6t_4$$

Every element in the central of this group is the commutator of an element of the commutator subgroup not in the central and an element $u_1^{y}u_2^{y}u_3^{y}u_4^{y}$ for the general element in C is $t_1^{a}t_2^{a}t_3^{a}t_4^{a}$ and we can show that for a given set of a_1, a_2, a_3, a_4 there exist sets of y_1, y_2, y_3, y_4 and x_1, x_2, x_3, x_4, x_5, x_6 such that

$$(u_4^{-y_4}\ldots u_1^{-y_1})(s_1^{x_1}\ldots s_6^{x_6})(u_1^{y_1}\ldots u_4^{y_4})=(s_1^{x_1}\ldots s_6^{x_6})(t_1^{a_1}\ldots t_4^{a_4}).$$

If we let $s_1^{x_1}\ldots s_6^{x_6}=s$, the left side becomes

$$s\,t_1^{x_4y_1-x_2y_2+x_1y_3}\,t_2^{x_5y_1-x_3y_2+x_1y_4}\,t_3^{x_6y_1-x_3y_3+x_2y_4}\,t_4^{x_6y_2-x_5y_3+x_4y_4}$$

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[3] *Ibid.*

[4] W. Dyck, "Über die Zusammensetzung einer Gruppe discreter Operationen, über ihre Primitivität und Transitivität," Mathematische Annalen, vol. 22 (1883), p. 76.

which must equal $s\ t_1^{a_1}\ t_2^{a_2}\ t_3^{a_3}\ t_4^{a_4}$ so that we are led to the equations

$$(1)\qquad \begin{cases} x_4y_1 - x_2y_2 + x_1y_3 = a_1 \\ x_5y_1 - x_3y_2 + x_1y_4 = a_2 \\ x_6y_1 - x_3y_3 + x_2y_4 = a_3 \\ x_6y_2 - x_5y_3 + x_4y_4 = a_4 \end{cases}$$

where not all the a_i are zero. Now this system of equations will have a solution for $y_1, \ldots, y_4$ if the matrix of coefficients is either of rank four or of the same rank as the augmented matrix. In the first case we have

$$D = \begin{vmatrix} x_4 & -x_2 & x_1 & 0 \\ x_5 & -x_3 & 0 & x_1 \\ x_6 & 0 & -x_3 & x_2 \\ 0 & x_6 & -x_5 & x_4 \end{vmatrix} \neq 0.$$

But $D=(x_1x_6-x_2x_5+x_3x_4)^2$. Thus for any set of a_1, a_2, a_3, a_4, we may choose arbitrarily a set of $x_1, \ldots, x_6$ such that

$$x_1x_6 - x_2x_5 + x_3x_4 \neq 0$$

which, in turn, will enable us to determine uniquely a set of $y_1, \ldots, y_4$ such that $t_1^{a_1}\ t_2^{a_2}\ t_3^{a_3}\ t_4^{a_4}$ is the commutator of $u_1^{y_1}\ u_2^{y_2}\ u_3^{y_3}\ u_4^{y_4}$ and $s_1^{x_1} \ldots s_6^{x_6}$.

The rank of the matrix of coefficients in the system of equations (1) is at least two since not all the x_i are zero. Furthermore the rank cannot be three since all three-rowed minors have determinants which are zero or are multiples of $x_1x_6-x_2x_5+x_3x_4$ which is necessarily zero if the rank is not four. We have considered the case where the rank is four, so let us now suppose the rank to be two. If the system (1) is to have a solution in this case, the augmented matrix

$$\overline{M} = \left\| \begin{matrix} x_4 & -x_2 & x_1 & 0 & a_1 \\ x_5 & -x_3 & 0 & x_1 & a_2 \\ x_6 & 0 & -x_3 & x_2 & a_3 \\ 0 & x_6 & -x_5 & x_4 & a_4 \end{matrix} \right\|$$

must have the same rank as the matrix of coefficients M, thus leading to the set of conditions

$$(2)\qquad \begin{cases} a_3x_1 - a_2x_2 + a_1x_3 = 0 \\ a_4x_1 - a_2x_4 + a_1x_5 = 0 \\ a_4x_2 - a_3x_4 + a_1x_6 = 0 \\ a_4x_3 - a_3x_5 + a_2x_6 = 0 \end{cases}$$

The rank of the matrix of coefficients

$$N = \left\| \begin{matrix} a_3 & -a_2 & a_1 & 0 & 0 & 0 \\ a_4 & 0 & 0 & -a_2 & a_1 & 0 \\ 0 & a_4 & 0 & -a_3 & 0 & a_1 \\ 0 & 0 & a_4 & 0 & -a_3 & a_2 \end{matrix} \right\|$$

is three. That it is at least three is evident since not all the a_i are zero.

That the rank cannot be greater than three follows from the linear dependence of the four rows with $-a_4$, a_3, $-a_2$ and a_1 as the constants of dependence. Thus we not only see that for any given set of $a_1, \ldots, a_4$ there exists a solution for $x_1, \ldots, x_6$ in the system (2), but we also see that only three of the solutions are independent. Any values of $x_1, \ldots, x_6$ which satisfy the system (2) also satisfy the condition

$$x_1x_6 - x_2x_5 + x_3x_4 = 0 \tag{3}$$

since the determinant of the matrix of coefficients of the system (2) considered as a set of four equations in $a_1, \ldots, a_4$ is

$$\begin{vmatrix} x_3 & -x_2 & x_1 & 0 \\ x_5 & -x_4 & 0 & x_1 \\ x_6 & 0 & -x_4 & x_2 \\ 0 & x_6 & -x_5 & x_4 \end{vmatrix} = (x_1x_6 - x_2x_5 + x_3x_4)^2$$

which must be zero when the system is consistent. Hence any solution of (2) will reduce the ranks of the matrix of coefficients M and the augmented matrix $\overline{M}$ of the system (1) to two, so that in the solution of the system (1) for $y_1, \ldots, y_4$, two of the values may be assigned at pleasure and the remaining two uniquely determined provided the rank of the matrix of the remaining two variables is two.

The situation may be summarized geometrically if we consider $x_1, \ldots, x_6$ and $y_1, \ldots, y_4$ as determining a projective 5-space X and 3-space Y respectively. Any element in the central of G is expressible either as the commutator of the elements corresponding to any point x in X not on the four dimensional manifold V defined by (3) and the unique point y in Y determined by the point x by means of the system of equations (1), or as the commutator of the elements corresponding to any point x in some plane π in X which is wholly on V and any point y of a line λ in Y, where λ is not independent of π.

Let $t = t_1^{a_1} t_2^{a_2} t_3^{a_3} t_4^{a_4}$ be any element of the central C. It generates the cyclic subgroup $\{t\}$ in C. Since t is the commutator of the elements $u = u_1^{y_1} \ldots u_4^{y_4}$ and $s = s_1^{x_1} \ldots s_6^{x_6}$, we may rename the generators of G in the following fashion to show that the quotient group $G/\{t\}$ is of the same type as $G/\{t_1\}$. We have $t = s^{-1}u^{-1}su$ where, according to the preceding discussion, we may always determine s for any t in such a manner that condition (3) is satisfied. In the case of the group of class two (i.e., where the second commutators are all the identity) generated by four operators where the only relations are $a_i^p = I$, for every element $a_i \neq I$ in the group, (3) is the necessary and sufficient condition that s be a commutator of v and w, where $v = u_1^{k_1} \ldots u_4^{k_4}$

and $w = u_1^{l_1} \ldots u_4^{l_4}$.[5] Hence if this condition is satisfied when G is of class 3, v and w may be determined so that $v^{-1}w^{-1}vw = s\bar{t}$ where $\bar{t}$ is some element in the central. Hence

$$t = (s\bar{t})^{-1}u^{-1}(s\bar{t})u = (v^{-1}w^{-1}vw)^{-1}u^{-1}(v^{-1}w^{-1}vw)u.$$

Then let $U_1 = v = u_1^{k_1} \ldots u_4^{k_4}$, $U_2 = w = u_1^{l_1} \ldots u_4^{l_4}$, and $U_3 = u = u_1^{y_1} \ldots u_4^{y_4}$. Choose $U_4 = u_1^{z_1} \ldots u_4^{z_4}$ in such a fashion that

$$\begin{vmatrix} k_1 & k_2 & k_3 & k_4 \\ l_1 & l_2 & l_3 & l_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix} \neq 0.$$

Then $S_1 = U_1^{-1} U_2^{-1} U_1 U_2 = s\bar{t}$ and $T_1 = S_1^{-1} U_3^{-1} S_1 U_3 = t$.
With this renaming of generators in G, $G/\{t\} \equiv G/\{T_1\}$.

Thus only one type of group of class 3 generated by 4 operators is obtained as the quotient group of G with respect to a subgroup of order 3 in the central. Such a group is $G/\{t_1\} \cong G_1$. G also contains but one type of invariant subgroup of each of the orders 3^2, 3^4 and 3^5 leading to new groups of class 3, while there are 2 types of subgroups of order 3^3; namely, $\{t_1, t_2, t_3\}$ contained in C and $\{s_1, t_1, t_2\}$ which is not in C. All the groups of class 3 generated by 4 operators are listed in the following table.

	Order	Central C	Commutator Sub-group K	G/C	Characteristic* Subgroups
G	3^{14}	3^4	3^{10}	3^{10} $C{:}3^6$, $K{:}3^6$	
$G/\{t_1\}$	3^{13}	3^3	3^9	3^{10} $C{:}3^6$, $K{:}3^6$	$\{K_1, u_1, u_2, u_3\}$, 3^{12} metabelian
$G/\{t_1, t_2\}$	3^{12}	3^3	3^8	3^9 $C{:}3^5$, $K{:}3^5$	$\{K_2, u_1, u_2\}$, 3^{10} metabelian, $C{:}3^7$, $K{:}3^3$ $\{K_2, u_3, u_4\}$, 3^{10} metabelian, $C{:}3^4$, $K{:}3^3$
$G/\{t_1, t_2, t_3\}$	3^{11}	3^4	3^7	3^7 $C{:}3^4$, $K{:}3^3$	$\{K_3, u_1\}$, 3^8 abelian
$G/\{s_1, t_1, t_2\}$	3^{11}	3^2	3^7	3^9 $C{:}3^5$, $K{:}3^5$	$\{K_4, u_1, u_2\}$, 3^9 metabelian
$G/\{s_1, t_1, t_2, t_3\}$	3^{10}	3^3	3^6	3^7 $C{:}3^4$, $K{:}3^3$	$\{K_5, u_1\}$, 3^7 abelian
$G/\{s_1, s_2, t_1, t_2, t_3\}$	3^9	3^2	3^5	3^7 $C{:}3^4$, $K{:}3^3$	$\{K_6, u_1\}$, 3^6 abelian

* Other than I, C, K, and the group itself.

[5] H. R. Brahana, "Finite Metabelian Groups and Plücker Line Coordinates," American Journal of Mathematics, vol. LXII, no. 2 (1940), p. 377.

3. The group G generated by 5 operators $u_1, \ldots, u_5$ all of whose elements $\neq I$ are of period 3 is of order 3^{25} with a central C of order 3^{10} contained in a commutator subgroup K of order 3^{20}. Again every element in C is the commutator of an element in K but not in C, and an element $u_1^{y_1} \ldots u_5^{y_5}$, but there are two types of commutators available in C.

Naming the commutators according to the system used in §2 any element $t = t_1^{a_1} \ldots t_{10}^{a_{10}}$ in C is the commutator of $u = u_1^{y_1} \ldots u_5^{y_5}$ and $s = s_1^{x_1} \ldots s_{10}^{x_{10}}$ if

$$(1)\quad \begin{cases} y_3x_1 - y_2x_2 + y_1x_5 = a_1 \\ y_4x_1 - y_2x_3 + y_1x_6 = a_2 \\ y_5x_1 - y_2x_4 + y_1x_7 = a_3 \\ y_4x_2 - y_3x_3 + y_1x_8 = a_4 \\ y_5x_2 - y_3x_4 + y_1x_9 = a_5 \\ y_5x_3 - y_4x_4 + y_1x_{10} = a_6 \\ y_4x_5 - y_3x_6 + y_2x_8 = a_7 \\ y_5x_5 - y_3x_7 + y_2x_9 = a_8 \\ y_5x_6 - y_4x_7 + y_2x_{10} = a_9 \\ y_5x_8 - y_4x_9 + y_3x_{10} = a_{10} \end{cases}$$

The rank of the matrix M of coefficients is 6. For any set of $a_1, \ldots, a_{10}$ there exists a set of $y_1, \ldots, y_5$ such that the augmented matrix $\overline{M}$ also has rank 6, thus insuring a solution for $x_1, \ldots, x_{10}$ in (1), but there are two types of solutions dependent upon the nature of the a_i. According to whether $a_1, \ldots, a_{10}$ is on or off the 6-dimensional manifold V defined by

$$(2)\quad \begin{cases} a_1a_6 - a_2a_5 + a_3a_4 = 0 \\ a_1a_9 - a_2a_8 + a_3a_7 = 0 \\ a_1a_{10} - a_4a_8 + a_5a_7 = 0 \\ a_2a_{10} - a_4a_9 + a_6a_7 = 0 \\ a_3a_{10} - a_4a_9 + a_6a_8 = 0 \end{cases}$$

t is the commutator of s and u where s itself is or is not a commutator. A commutator of the first type is t_1 corresponding to the point $a_1 = 1$, $a_i = 0$, $i = 2, \ldots, 10$, which is obviously on V. t_1 is the commutator of u_1 and s_5, where s_5 is the commutator of u_3 and u_4. t_1t_6 corresponding to the point 1,0,0,0,0,1,0,0,0,0 not on V is the commutator of u_1 and s_5s_{10} which is not a commutator. The two kinds of groups of class 3 obtained as quotient groups of G with respect to subgroups of order 3 in C are typified by $G/\{t_1\}$ and $G/\{t_1t_6\}$ both of which are generated by 5 operators, are of order 3^{24} with a central of order 3^9 and a commutator subgroup of order 3^{19}. If the commutator subgroup of $G/\{t_1\}$ is denoted by K_1, $G/\{t_1\}$ contains a metabelian subgroup $\{K_1, u_1, u_2, u_3\}$ of order 3^{22}, in contrast to $G/\{t_1t_6\}$ all of whose subgroups of this order are of class 3.

A complete determination of all the groups of class 3 generated by 5 operators would depend upon the investigation of the geometry of the manifold V in the projective 19-space determined by $s_1, \ldots, s_{10}$, $t_1, \ldots, t_{10}$.

4. The situation is somewhat different in the case of 6 generators where G is of order 3^{41} with a central C of order 3^{20} and a commutator subgroup K of order 3^{35}. Here there are at least 3 types of subgroups of order 3 in the central. The system of equations corresponding to (1) in §3 is such that there may be no solution, or solutions of the two types found in the case of five generators, depending upon the nature of $a_1, \ldots, a_{20}$. Thus here for the first time there appear elements in C which are not commutators, so that there will be a new type of group of class 3 among the quotient groups $G/\{t\}$.

The groups $G_1 \cong G/\{t_1\}$ and $G_2 \cong G/\{t_{10}t_{11}\}$ are examples of two different kinds of groups of class three, since t_1 is the commutator of u_1 and s_1, while $t_{10}t_{11}$ cannot be expressed as the commutator of any u and s. Both of these groups are of order 3^{40} with a central of order 3^{19} and a commutator subgroup of order 3^{34}. G_1 contains a characteristic subgroup $\{K_1, u_1, u_2, u_3\}$ of order 3^{37} which is metabelian, while every subgroup of order 3^{37} in G_2 is of class 3. $G_3 \cong G/\{t_1t_8\}$ is an example of the third type of quotient group available. t_1t_8 is a commutator of u and s, where s is not a commutator. G_3, like G_2, contains only subgroups of class 3 of order 3^{37}. In G_2 the subgroups $\{u_1, u_5, u_6\}$ and $\{u_2, u_3, u_4\}$ are both of order 3^7, class 3, and have the same central, while the smallest group containing them is G_2 itself. In the case of G_3, the corresponding two subgroups of order 3^7 which are $\{u_1, u_2, u_3\}$ and $\{u_1, u_4, u_5\}$ also have the same central, but they are contained in a subgroup of index p in G_3.

VITA

Jewell Emma Schubert was born in Chicago, Illinois, on September 14, 1915. After attending the Chicago public schools, she enrolled at Northwestern University, where she was graduated with the degree of Bachelor of Science in June, 1936. After a year of graduate work at that institution, she went to Bryn Mawr College as a Resident Scholar in September, 1937. Following an apprenticeship of a year in the teaching of secondary school mathematics at the Baldwin School in Bryn Mawr, Pennsylvania, she taught at Miss Fine's School in Princeton, New Jersey, from September, 1939, until June, 1942, and studied at the University of Michigan during the summer of 1940. In September of 1942 she came to the University of Illinois as a fellow in mathematics, remained as an assistant during the following two years, was granted the degree of Master of Science in February, 1944, and awarded a fellowship in September, 1945. Honorary and professional organizations of which she is a member are Phi Beta Kappa, Phi Kappa Phi, Sigma Delta Epsilon and Pi Mu Epsilon.